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Vacuum configurations for renormalizable non-commutative scalar models

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Abstract. In this paper we find non-trivial vacuum states for the renormalizable non-commutative ϕ^4 model. An associated linear sigma model is then considered. We further investigate the corresponding spontaneous symmetry breaking.

1 Introduction

Non-commutative quantum field theory (for a general review, see [1, 2]) – that is, field theory based on non-commutative geometry (see [3] for a general review) – is nowadays one of the most appealing candidates for new physics beyond the standard model. Moreover, non-commutative field theory can be seen as an effective regime of string theory [4, 5]. From a different point of view, non-commutativity is well adapted for the description of physics in the presence of a background field, like for example the fractional quantum Hall effect [6–8].

Non-commutative physics is known to suffer from a new type of divergence, UV–IR mixing; a divergence that is responsible for the non-renormalizability of the models. This difficulty was resolved for scalar Φ^4 models by the introduction of a new harmonic term in the action – the Grosse–Wulkenhaar model. The model was proven to be renormalizable at any order in perturbation theory [9–13]. Moreover, the parametric representation was introduced [14] and then dimensional regularization and renormalization were performed [15]. Let us also emphasize here that the Hopf algebra description of this type of renormalization was given in [16]. Moreover, let us also stress here the fact that it was recently shown [17] that this type of action can be interpreted from the spectral action point of view (for the latest developments, see [18]).

The Grosse–Wulkenhaar model was however proven to have a better flow behavior with respect to the commutative ϕ^4 model. Indeed, in [19–21] was proven that this model does not present a Landau ghost; let us recall that this was not the case for the commutative model.

Another improvement with respect to commutative scalar quantum field theory is that a constructive version (for a general review, see [22]) is within reach [23, 24].

In this paper we first obtain vacuum states which, because of the presence of this new harmonic term, must be non-trivial functions of the space-time position x. Note that a somewhat similar conclusion regarding a non-constant vacuum was also obtained recently in [17]. We then analyze the issue of spontaneous symmetry breaking for a non-commutative analog of the linear sigma model with a harmonic term at the classical level. The model we consider here is a non-commutative linear sigma model based on a set of N scalar fields but in the presence of harmonic terms for each of these scalars.

Note that the Goldstone theorem for the non-commutative linear sigma model without harmonic term was already investigated up to one loop [25–27] and two loops [28]. Other investigations regarding different non-commutative models were done in [29–32]. Within these models it was found that the situation for the Goldstone theorem followed rather closely the features of the commutative case. In the present case we find that the situation is much more involved.

The paper is organized as follows. In Sect. 2 we give some notation and conventions and we introduce the Grosse–Wulkenhaar model as well as some existing results. In Sect. 3 we find vacuum states $v(x)$, analyzing under what conditions they are solutions of the equations of motion. Then we investigate the issue of spontaneous symmetry breaking for the linear sigma model. Finally, in Sect. 4 our concluding remarks and discussions are given.

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2 Notation and conventions: the Grosse–Wulkenhaar model

We first collect the basic ingredients on the Moyal algebras (see for example [33–37] and references therein). We consider a D-dimensional Moyal algebra $\mathcal{M}$ that can conveniently be defined using the following relation:

$$[x^\mu, x^\nu]_\star = \mathrm{i}\Theta^{\mu\nu}\,, \tag{1}$$

where $[a,b]_\star = a\star b - b\star a$ and the skewsymmetric matrix Θ is given by

$$\Theta = \begin{pmatrix} 0 & -\theta & & & \\ \theta & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & -\theta \\ & & & \theta & 0 \end{pmatrix}. \tag{2}$$

The Moyal product of two functions f and g can be defined by

$$(f\star g)(x) = \frac{1}{\pi^D|\det\Theta|}\int \mathrm{d}^Dy\,\mathrm{d}^Dz\, f(x+y)g(x+z)\mathrm{e}^{-2\mathrm{i}y\Theta^{-1}z}\,. \tag{3}$$

We will mainly consider the cases $D=2$ and $D=4$. Let us now list some useful formulas, which will be used in the calculations. The trace and cyclicity relations are given by

$$\int \mathrm{d}^Dx f(x)\star g(x) = \int \mathrm{d}^Dx f(x)g(x)\,,$$
$$\int \mathrm{d}^Dx f(x)\star g(x)\star h(x) = \int \mathrm{d}^Dx h(x)\star f(x)\star g(x)\,, \tag{4}$$

for any functions f, g and h. Furthermore let

$$\widetilde{x} = 2\Theta^{-1}x\,. \tag{5}$$

Note that this vector is known to play a crucial role in the construction of canonical gauge invariant connections [37–40]. One has

$$\partial_\mu\phi = -\frac{\mathrm{i}}{2}[\widetilde{x}_\mu,\phi]_\star\,,$$
$$\widetilde{x}_\mu\phi = \frac{1}{2}\{\widetilde{x}_\mu,\phi\}_\star\,, \tag{6}$$

where $\{a,b\}_\star = a\star b + b\star a$. Note that (2) and (5) lead to

$$\partial^\mu\widetilde{x}_\mu = 0 \tag{7}$$

and that (6) can be rewritten as

$$\widetilde{x}_\mu\star f = \widetilde{x}_\mu f + \mathrm{i}\partial_\mu f\,,$$
$$f\star\widetilde{x}_\mu = \widetilde{x}_\mu f - \mathrm{i}\partial_\mu f\,. \tag{8}$$

The Moyal space, being a linear space of infinite dimension, admits a particular basis, the matrix basis (for more details, see for example [33, 41] and references therein). This basis involves an infinite set of Schwartz functions, which for $D=2$ can be indexed by two natural numbers m and n, namely $f_{mn}(x)$. Some relevant properties are given in the appendix.

We consider the Euclidean action for the Grosse–Wulkenhaar model [9, 10] and its complex-valued version, respectively given by

$$S[\phi] = \int \mathrm{d}^Dx \left(\frac{1}{2}\partial_\mu\phi\star\partial_\mu\phi + \frac{\Omega^2}{2}(\widetilde{x}_\mu\phi)\star(\widetilde{x}_\mu\phi) - \frac{\mu^2}{2}\phi\star\phi + \lambda\,\phi\star\phi\star\phi\star\phi\right) \tag{9}$$

and

$$S_C[\phi] = \int \mathrm{d}^Dx \left(\partial_\mu\phi^\dagger\star\partial_\mu\phi + \Omega^2(\widetilde{x}_\mu\phi)^\dagger\star(\widetilde{x}_\mu\phi) - \mu^2\phi^\dagger\star\phi + \lambda\,\phi^\dagger\star\phi\star\phi^\dagger\star\phi\right)\,, \tag{10}$$

where one considers a negative mass parameter, $-\mu^2$.

Let us emphasize here that the interest in the analysis of a real field ϕ comes basically from the possible insights with respect to the study of non-commutative gauge theories. Recently, potential candidates for renormalizable gauge theories on Moyal spaces have been singled out in [38, 39] (see also [37, 40]). Although such a construction is a necessary step towards the elaboration of renormalizable gauge theory, it has been recognized that these candidates have a non-trivial vacuum [17, 38, 39]. Its explicit determination is the next challenging problem that must be overcome in order to build a meaningful perturbative expansion that could be used to check renormalizability.

Recall that these actions are covariant under the Langmann–Szabo duality [42] duality, which relates the IR and UV regions. Throughout this paper we mainly assume $\Omega=1$ (the point where the actions (9) and (10) become invariant under this Langmann–Szabo duality). Moreover, the value $\Omega=1$ is stable under the flows of the renormalization group [19–21].

In the following we will determine and study a class of non-trivial minima for the real-valued field theory (9), using the matrix space formalism. It turns out that non-trivial (i.e. non-constant) vacua occur generically for (9) and (10) whenever $\Omega\neq 0$.

3 Vacuum configurations; spontaneous symmetry breaking

3.1 The complex-valued scalar field theory

Let us first treat the case of the complex field. The equation of motion obtained from (10) is

$$-\partial^2\phi + \Omega^2\widetilde{x}^2\phi - \mu^2\phi + 2\lambda\phi\star\phi^\dagger\star\phi = 0\,. \tag{11}$$

From (11) it can easily be seen that the harmonic term prevents a constant non-zero field from satisfying the equation of motion. This implies that constant vacua are forbidden, contrary to what happens in commutative models as well as in non-commutative models with $\Omega = 0$. Note that, as already stated in the introduction, a similar observation was done in [17]. Moreover, note that a specific type of non-constant configuration $v(x)$, stripe phases, were analyzed in a different context in a non-commutative framework in [30].

Using (6), (11) can be rewritten as

$$\frac{1}{4}\left(1+\Omega^2\right)\left(\widetilde{x}^2\star\phi+\phi\star\widetilde{x}^2\right) - \frac{1}{2}\left(1-\Omega^2\right)\widetilde{x}_\mu\star\phi\star\widetilde{x}_\mu - \mu^2\phi + 2\lambda\phi\star\phi^\dagger\star\phi = 0\,. \quad (12)$$

Specializing from now on to the case $\Omega = 1$, the equation of motion simplifies to

$$\frac{1}{2}\left(\widetilde{x}^2\star\phi+\phi\star\widetilde{x}^2\right) - \mu^2\phi + 2\lambda\phi\star\phi^\dagger\star\phi = 0\,. \quad (13)$$

For reasons of simplicity, we restrict ourselves to $D = 2$. The adaptation of this analysis to the case $D = 4$ is straightforward, as will be explained in the sequel.

We now look for solutions of (13) in the form

$$v(x) = a f_{m_0 n_0}(x)\,, \quad (14)$$

where m_0, n_0 are fixed integers, $a \in \mathbb{C}^*$ and $f_{m_0 n_0}(x)$ are elements of the matrix basis (see the appendix). The equation of motion (13) is written in the matrix basis

$$\frac{4}{\theta}(m+n+1)\phi_{mn} - \mu^2\phi_{mn} + 2\lambda\,\phi_{mk}\phi^\dagger_{kl}\phi_{ln} = 0\,. \quad (15)$$

Inserting now in (15) the ansatz (14), written as

$$\phi_{mn} = a\delta_{mm_0}\delta_{nn_0}\,, \quad a \in \mathbb{C}^* \quad (16)$$

one has

$$a\left(\frac{4}{\theta}(m_0+n_0+1) - \mu^2 + 2\lambda|a|^2\right) = 0\,. \quad (17)$$

This implies

$$|a|^2 = \frac{1}{\lambda\theta}\left(\frac{\mu^2\theta}{2} - 2(m_0+n_0+1)\right), \quad (18)$$

so that consistency requires the following condition on the mass (or equivalently on the indices m_0 and n_0):

$$\mu^2 > \frac{4}{\theta}(m_0+n_0+1)\,. \quad (19)$$

Thus all the functions proposed in (14) with a satisfying (18) are solutions of the equation of motion if the condition (19) is verified. We denote

$$p_C = \left\lfloor \frac{\mu^2\theta}{4} - 1 \right\rfloor$$

(where $\lfloor . \rfloor$ is the integer part). If p_C is negative, the only possibility is the trivial solution. If p_C is positive, the solutions denoted by the indices m_0 and n_0 have to satisfy the constraint $m_0 + n_0 \le p_C$. The number of solutions of the form (14) is $\sum_{k=0}^{p_C}(p_C - k + 1) = \frac{(p_C+1)(p_C+2)}{2}$. We will proceed with the interpretation of these observations in the next subsection, where a similar description can be done.

At this point, we make the following observation. The transformation

$$\begin{aligned} &\phi_{m,n} \mapsto \phi'_{m,n} = \phi_{m-1,n} \quad (\forall n \in \mathbb{N}\,, \quad \phi'_{0,n} = 0) \\ &\mu^2 \mapsto \mu^2 + \frac{4}{\theta} \end{aligned} \quad (20)$$

is a symmetry of the equation of motion (13). Let us take a solution $v_{mn} = a(\mu^2)\delta_{mm_0}\delta_{nn_0}$ and let this symmetry act on it: $v'_{m,n} = a\left(\mu^2 - \frac{4}{\theta}\right)\delta_{m,m_0+1}\delta_{n,n_0}$; then $v'_{m,n}$ is also a solution of the equation of motion. As the complex conjugation is also a symmetry of (13), we find that the composition

$$\begin{aligned} &\phi_{m,n} \mapsto \phi'_{m,n} = \phi_{m,n-1} \quad (\forall m \in \mathbb{N}\,, \quad \phi'_{m,0} = 0)\,, \\ &\mu^2 \mapsto \mu^2 + \frac{4}{\theta} \end{aligned} \quad (21)$$

is also a symmetry. We notice that with these two transformations, all the solutions (14) of the equation of motion can be derived from a single one, $\sqrt{\frac{2}{\lambda\theta}\left(\frac{\mu^2\theta}{4} - 1\right)}f_{00}(x)$.

For $\Omega \neq 1$, the equation of motion (12) is written in the matrix basis

$$\begin{aligned} &\frac{2}{\theta}(1+\Omega^2)(m+n+1)\phi_{mn} \\ &- \frac{2}{\theta}(1-\Omega^2)\sqrt{(m+1)(n+1)}\phi_{m+1,n+1} \\ &- \frac{2}{\theta}(1-\Omega^2)\sqrt{mn}\phi_{m-1,n-1} - \mu^2\phi_{mn} + 2\lambda\,\phi_{mk}\phi^\dagger_{kl}\phi_{ln} = 0\,. \end{aligned} \quad (22)$$

All solutions of the form (14) do not verify this new equation (22) if $\Omega \neq 1$. Nevertheless, it is possible to find some solutions of (22), for instance the sum of two elements of the matrix basis. The full analysis of the case $\Omega \neq 1$ deserves further investigation, which goes beyond the scope of this paper.

3.2 The real-valued scalar field theory

In the case of a real field ϕ, the equation of motion derived from (9) reads

$$-\partial^2\phi + \Omega^2\widetilde{x}^2\phi - \mu^2\phi + 4\lambda\phi\star\phi\star\phi = 0\,, \quad (23)$$

which for $\Omega = 1$ can be rewritten as

$$\frac{1}{2}(\widetilde{x}^2\star\phi+\phi\star\widetilde{x}^2) - \mu^2\phi + 4\lambda\phi\star\phi\star\phi = 0\,. \quad (24)$$

We now look for solutions of (24) whose form is given by a similar ansatz to the one of the complex case; see (14). Note, however, that the vacuum must now be consistent with the reality condition $v^\dagger(x) = v(x)$. We put

$$v(x) = a_{m_0} f_{m_0 m_0}(x)\,, \quad m_0 \in \mathbb{N}\,. \quad (25)$$

Note that there is no Einstein summation convention (m_0 is fixed). The equation of motion (24) can be re-expressed in the matrix basis:

$$\frac{4}{\theta}(m+n+1)\phi_{mn} - \mu^2\phi_{mn} + 4\lambda\,\phi_{mk}\phi_{kl}\phi_{ln} = 0\,, \qquad (26)$$

and the ansatz

$$\phi_{mn} = a_{m_0}\delta_{mm_0}\delta_{nm_0}, \quad a_{m_0} \in \mathbb{R}^*\,. \qquad (27)$$

By insertion of (27) in (26), one finds

$$a_{m_0}\left(\frac{4}{\theta}(2m_0+1) - \mu^2 + 4\lambda a_{m_0}^2\right) = 0\,. \qquad (28)$$

As a consequence, a_{m_0} has to satisfy

$$a_{m_0}^2 = \frac{1}{\lambda\theta}\left(\frac{\mu^2\theta}{4} - 2m_0 - 1\right) \qquad (29)$$

and the mass

$$\mu^2 > \frac{4}{\theta}(2m_0+1)\,. \qquad (30)$$

Now, upon setting

$$p = \left\lfloor \frac{\mu^2\theta}{8} - \frac{1}{2} \right\rfloor \qquad (31)$$

(the counterpart of p_C for the real-valued theory), the discussion proceeds along the same lines as in the previous subsection. Namely, if p is negative, no index can satisfy the condition (30). If p is positive, there are $(p+1)$ solutions of the form (25) satisfying the constraint (30). Notice that the counterpart of the symmetries (20) and (21) is now

$$\begin{aligned} &\phi_{m,n} \mapsto \phi'_{m,n} = \phi_{m-1,n-1} \\ &\quad (\forall m,n \in \mathbb{N}\,, \quad \phi'_{0,n} = 0\,, \quad \phi'_{m,0} = 0)\,, \\ &\mu^2 \mapsto \mu^2 + \frac{8}{\theta}\,, \end{aligned} \qquad (32)$$

so that all the vacua of the form (25) are derived from $\sqrt{\frac{2}{\lambda\theta}\left(\frac{\mu^2\theta}{8}-\frac{1}{2}\right)}f_{00}(x)$.

Let us now look for more general solutions of type

$$v(x) = \sum_{k=0}^{\infty} a_k f_{kk}(x)\,, \quad (a_k) \in \mathbb{R}^{\mathbb{N}}\,. \qquad (33)$$

The equation of motion (24) then leads to the following condition on the coefficients a_k:

$$a_k = 0 \quad \text{or} \quad a_k^2 = \frac{1}{\lambda\theta}\left(\frac{\mu^2\theta}{4} - 2k - 1\right). \qquad (34)$$

Owing to the analysis given above, one readily infers that the sum involved in (33) cannot run to infinity simply because one must have $k \le p$.

Therefore, $\sum_{k=0}^{p} a_k f_{kk}(x)$ represents a new set of solutions for the equation of motion, which cannot be derived from a single one under the symmetry (32). One now has for the coefficients of $v\star v\star\phi$ and $v\star\phi\star v$:

$$\begin{aligned} (v\star v\star\phi)_{mn} &= \left(\sum_{k=0}^{p} a_k^2\delta_{mk}\right)\phi_{mn}\,, \\ (v\star\phi\star v)_{mn} &= \left(\sum_{k,l=0}^{p} a_k a_l \delta_{mk}\delta_{nl}\right)\phi_{mn}\,. \end{aligned} \qquad (35)$$

Let us now check if these solutions are minima for the action. The quadratic part of the action (9) reads

$$\begin{aligned} S_{\text{quadr}} = 2\pi\theta\Bigg(&\frac{2}{\theta}(m+n+1) - \frac{\mu^2}{2} + 2\lambda\sum_{k=0}^{p} a_k^2(\delta_{mk}+\delta_{nk}) \\ &+ 2\lambda\sum_{k,l=0}^{p} a_k a_l \delta_{mk}\delta_{nl}\Bigg)\phi_{nm}\phi_{mn}\,. \end{aligned} \qquad (36)$$

The propagator $C_{mn,kl}$ is diagonal:

$$C_{mn,kl} = C_{mn}\delta_{ml}\delta_{nk}\,.$$

In order to have a minimum, one thus needs C_{mn} to be positive for all $m,n \in \mathbb{N}$. From (36) one has

$$\begin{aligned} C_{mn}^{-1} = \alpha_{mn} &+ 4\lambda\pi\theta\sum_{k=0}^{p} a_k^2(\delta_{mk}+\delta_{nk}) \\ &+ 4\lambda\pi\theta\sum_{k,l=0}^{p} a_k a_l \delta_{mk}\delta_{nl}\,, \end{aligned} \qquad (37)$$

where

$$\alpha_{mn} = 4\pi(m+n+1) - \mu^2\pi\theta\,, \quad \text{and} \quad a_k^2 = 0 \quad \text{or} \quad a_k^2 = -\frac{\alpha_{kk}}{4\lambda\pi\theta}\,. \qquad (38)$$

For $p \ge 0$, one has to distinguish between the following cases.

- $m > p$ and $n > p$:

$$C_{mn}^{-1} = \alpha_{mn} = 4\pi\left(m+n-2\left(\frac{\mu^2\theta}{8}-\frac{1}{2}\right)\right) > 0\,. \qquad (39)$$

- $m \le p$ and $n > p$: if $a_m^2 = 0$ then

$$C_{mn}^{-1} = \alpha_{mn} = 4\pi\left(m+n-2\left(\frac{\mu^2\theta}{8}-\frac{1}{2}\right)\right). \qquad (40)$$

  In order not to have for a certain value of n ($p < n \le 2p-m$) that $C_{mn}^{-1} < 0$, one needs to have $a_m^2 = -\frac{\alpha_{mm}}{4\lambda\pi\theta}$. In this case, we have

$$C_{mn}^{-1} = \alpha_{mn} - \alpha_{mm} = 4\pi(n-m) > 0\,. \qquad (41)$$

- $m > p$ and $n \le p$: is treated along the same lines as above. $C_{mn}^{-1} = 4\pi(m-n) > 0$, because we assumed that $\forall k \in \{0,...,p\}$, $a_k^2 = -\frac{\alpha_{kk}}{4\lambda\pi\theta}$.

– $m \leq p$ and $n \leq p$:

$$C^{-1}_{mn} = \alpha_{mn} - \alpha_{mm} - \alpha_{nn} + \sqrt{\alpha_{mm}\alpha_{nn}} \\ = -\alpha_{mn} + \sqrt{\alpha_{mm}\alpha_{nn}} \geq 0\,, \quad (42)$$

and $C^{-1}_{mn} = 0$ holds if and only if $m = n = \frac{\mu^2\theta}{8} - \frac{1}{2} \in \mathbb{N}$.

From the above analysis, one can conclude that one has a positive defined propagator just for a single solution. This solution, which is a minimum of the action (9), corresponds to

$$v(x) = \sum_{k=0}^{p} a_k f_{kk}(x)\,, \quad (43)$$

where $a_k^2 = \frac{1}{\lambda\theta}\left(\frac{\mu^2\theta}{4} - 2k - 1\right)$.

For $\phi(x) = \sum_{k,l=0}^{\infty} \phi_{mn} f_{mn}(x)$, a solution of the equation of motion (24), we compute the value of the action (9) in the matrix basis:

$$S[\phi] = 2\pi \sum_{k,l=0}^{\infty} \left(m + n - 2\left(\frac{\mu^2\theta}{8} - \frac{1}{2}\right)\right) |\phi_{mn}|^2\,. \quad (44)$$

So for $p < 0$,

$$S[\phi] \geq S[0] = 0\,, \quad (45)$$

and for $p \geq 0$ and $v(x)$ the vacuum (43),

$$S[v] = -\frac{8\pi}{\lambda\theta} \sum_{k=0}^{p} \left(\frac{\mu^2\theta}{8} - \frac{1}{2} - k\right)^2 < 0\,. \quad (46)$$

At this point, a remark is to be made. In commutative QFT or in non-commutative QFT without the harmonic term, one has the phenomenon of spontaneous symmetry breaking as soon as the mass parameter is taken to be negative. For the models considered here this is not the case anymore. Indeed, if the mass parameter does not go beyond a certain limit $\mu^2 = \frac{4}{\theta}$, then from (45), we see that $\phi(x) = 0$ is the global minimum of the action (9), as p is negative. So the harmonic term will prevent the phenomenon of spontaneous symmetry breaking from occurring.

When the mass parameter exceeds this critical value (i.e. $p \geq 0$), one can see from (40) that $\phi(x) = 0$ is no longer a local minimum of the action. Therefore one has to consider a non-trivial vacuum $v(x)$, as for example (43). Note that (43) corresponds to a different solution when changing the value of the limit parameter p. Let us now stress some of the features of these vacuum configurations. Owing again to the properties of the matrix basis (see again the appendix), it can be realized that (43) does not vanish for $x = 0$, while it decays at infinity (as a *finite* linear combination of the Schwartz functions $f_{mn}(x)$).

Let us further indicate that these results extend in $D = 4$ also. Indeed, one just has to replace the indices m by $(m_1, m_2) \in \mathbb{N}^2$ and the number m by $m_1 + m_2$. In four dimensions, it is well known that when computing radiative corrections, the mass parameter of a scalar field becomes huge (because of the quadratic divergence). In order to get a low value for the renormalized mass, one may thus consider a non-commutative scalar field theory with a harmonic term, a negative mass term and a non-trivial vacuum. It is further possible to choose (43) as this non-trivial vacuum, since the special value $\Omega = 1$ is stable under the renormalization group.

Moreover we also exhibit in the next subsection the following class of solutions of the equation of motion. If one has a configuration $v(x)$ that satisfies

$$v \star v = -\frac{1}{4\lambda}\tilde{x}^2 + \frac{\mu^2}{4\lambda} \quad (47)$$

(see (59), (60) and (63)) then $v(x)$ will also be a solution of the equation of motion. As already stated in Sect. 3.3 this type of equation can be easily shown to have non-trivial solutions using again the matrix basis.

We have thus pointed out in this subsection the existence of a non-trivial vacuum $v(x)$. Such a vacuum can be used in the next section, where we consider fluctuations of fields around such $v(x)$.

3.3 Spontaneous symmetry breaking for the linear sigma model

As a warming up issue, consider again the action (9) for a real ϕ field. This action has a discrete symmetry:

$$\phi \to -\phi\,. \quad (48)$$

As usual, assume that the system is near one of its minima $v(x)$. Upon setting

$$\phi(x) = v(x) + \sigma(x) \quad (49)$$

in (9), one obtains the Lagrangian in terms of the σ field

$$S = \int \mathrm{d}^4x \left(\left(\frac{1}{2}\partial_\mu\sigma\right) \star (\partial_\mu\sigma) + \frac{\Omega^2}{2}\tilde{x}^2\sigma\sigma - \frac{\mu^2}{2}\sigma\sigma \\ + 4\lambda v \star v \star \sigma \star \sigma + 2\lambda v \star \sigma \star v \star \sigma \\ + 4\lambda v \star \sigma \star \sigma \star \sigma + \lambda\sigma \star \sigma \star \sigma \star \sigma\right) \quad (50)$$

(where we have used (4)). Note that, as in the commutative case, the symmetry (48) has disappeared. The situation is exactly the same in the case of a complex field (10).

We now consider the linear sigma model built from the renormalizable scalar action (9), assuming again $\Omega = 1$. Additional considerations for $\Omega \neq 1$ as well as for the case of a complex-valued field will be given at the end of this section.

The action involves N valued fields ϕ_i and is given by

$$S_\sigma = \int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu\phi_i) \star (\partial_\mu\phi_i) + \frac{1}{2}\tilde{x}^2\phi_i\phi_i \\ - \frac{\mu^2}{2}\phi_i\phi_i + \lambda\phi_i \star \phi_i \star \phi_j \star \phi_j\right)\,. \quad (51)$$

The action above is invariant under the action of the orthogonal group $O(N)$ (as is also the case in the absence of the harmonic term [25–28].

Let

$$\langle \Phi \rangle = (0, \dots, 0, v(x)) \tag{52}$$

be a non-zero vacuum expectation value, where $v(x)$ is some minimum obtained from the equation of motion, as analyzed in the previous section.

Then shifting Φ to $\langle \Phi \rangle + \delta\Phi$ with

$$\delta\Phi = (\pi_1, \dots, \pi_{N-1}, \sigma(x)), \tag{53}$$

one obtains from (51)

$$\begin{aligned} S_\sigma = \int \mathrm{d}^4x \Big(&\frac{1}{2}(\partial_\mu \pi_i) \star (\partial_\mu \pi_i) + \frac{1}{2}\widetilde{x}^2 \pi_i \pi_i - \frac{\mu^2}{2}\pi_i\pi_i \\ &+ 2\lambda v \star v \star \pi_i \star \pi_i + \lambda \pi_i \star \pi_i \star \pi_j \star \pi_j \\ &+ \frac{1}{2}(\partial_\mu \sigma) \star (\partial_\mu \sigma) + \frac{1}{2}\widetilde{x}^2 \sigma\sigma - \frac{\mu^2}{2}\sigma\sigma \\ &+ 4\lambda v \star v \star \sigma \star \sigma + 2\lambda v \star \sigma \star v \star \sigma \\ &+ 2\lambda \sigma \star v \star \pi_i \star \pi_i + 2\lambda v \star \sigma \star \pi_i \star \pi_i \\ &+ 2\lambda \sigma \star \sigma \star \pi_i \star \pi_i + 4\lambda v \star \sigma \star \sigma \star \sigma \\ &+ \lambda \sigma \star \sigma \star \sigma \star \sigma \Big). \end{aligned} \tag{54}$$

Consider closer the part of the action (54) quadratic in the π fields:

$$\int \mathrm{d}^4x \left(\frac{1}{2}\widetilde{x}^2 \pi_i\pi_i - \frac{\mu^2}{2}\pi_i\pi_i + 2\lambda v \star v \star \pi_i \star \pi_i \right). \tag{55}$$

In the absence of the harmonic term, the linear sigma model supports a constant non-zero vacuum configuration leading to the appearance of N massless fields π. Indeed, the second and the third term in (55) balance each other. This is an obvious analog of the Goldstone theorem at the classical level, which has been further verified to the one- and respectively two-loop order in [25–27] and respectively [28].

When the harmonic term is included in (55) the situation changes substantially. Indeed, in view of the discussion for the scalar field theory presented above, constant non-zero vacuum configurations are no longer supported by the action. Thus, the cancellation of the π mass term does not occur automatically and must be reconsidered carefully. The attitude we adopt here is to mimic one of the main features of the Goldstone theorem of commutative field theory.

We thus investigate whether or not one can find a vacuum $v(x)$ that entails

$$\begin{aligned} &\int \mathrm{d}^4x \left(\frac{1}{2}\widetilde{x}^2 \pi_i\pi_i - \frac{\mu^2}{2}\pi_i\pi_i + 2\lambda v \star v \star \pi_i \star \pi_i \right) \\ &= \int \mathrm{d}^4x \left(\frac{1}{2}\widetilde{x}^2 \pi_i\pi_i - \frac{\mu^2}{2}(\pi_i \star \pi_i) + 2\lambda (v \star v)(\pi_i \star \pi_i) \right) \\ &= \int \mathrm{d}^4x \left(\frac{\Omega'^2}{2}\widetilde{x}^2 \pi_i\pi_i + \dots \right) \end{aligned} \tag{56}$$

Note that by the dots on the RHS of (56) we mean some eventual kinetic terms. If such a statement holds this means that one just has some harmonic type of term for the fields π; moreover, these fields would be non-massive. It is this what we propose as a corresponding Goldstone theorem in our case. Furthermore we also allow for the possibility $\Omega' = 0$ (the fields π are non-massive and they have no harmonic term either).

Moreover, note that this type of masslessness constraint for the π fields is the most general one that one can impose here (also introducing a harmonic like free parameter Ω').

Finally, looking at the LHS of the second line of (56) one sees that all the terms contain a $\pi \star \pi$ product except for the first one. In order to be able to factorize this $\pi \star \pi$ product, we now re-express the first term of the LHS also.

Using (4) and (8) one has

$$\begin{aligned} \int \mathrm{d}^4x \widetilde{x}^2 \pi_i\pi_i &= \int \mathrm{d}^4x (\widetilde{x}^2 \pi_i) \star \pi_i \\ &= \int \mathrm{d}^4x \left(\widetilde{x}^2 (\pi_i \star \pi_i) - \partial^\mu \pi_i \partial_\mu \pi_i - 2i\pi_i \widetilde{x}_\mu \partial_\mu \pi_i \right). \end{aligned} \tag{57}$$

Using now (7) one cancels out the last term in (57). All this becomes

$$\int \mathrm{d}^4x \widetilde{x}^2 \pi_i\pi_i = \int \mathrm{d}^4x \left(\widetilde{x}^2 (\pi_i \star \pi_i) - \partial^\mu \pi_i \partial_\mu \pi_i \right). \tag{58}$$

Note that this way of writing may be misleading in the sense that the kinetic term of the π fields seems to cancel with the second term in the RHS of (58). However this is just because of the particular way (58) of writing down the harmonic term at $\Omega = 1$.

One now introduces (58) in (56). Writing in the same way as above the RHS of (56) and leaving aside the kinetic terms, one is finally able to factorize the product $\pi \star \pi$ to get the following constraint for the vacuum v:

$$v \star v = -\frac{\omega^2}{4\lambda}\widetilde{x}^2 + \frac{\mu^2}{4\lambda}, \tag{59}$$

where

$$\omega^2 = 1 - \Omega'^2. \tag{60}$$

We have thus proven that this constraint is equivalent to the constraint (56) that we imposed on the kinetic terms of the π fields. Moreover, note that (59) presents non-trivial solutions $v(x)$ as can be seen for example in the matrix basis.

We now prove that a non-trivial vacuum $v(x)$ satisfying (59) is a solution of the equation of movement (24) if and only if one has $\Omega' = 0$. Indeed, inserting (59) in (24) one has

$$\begin{aligned} &\frac{1}{2}\left(\widetilde{x}^2 \star v + v \star \widetilde{x}^2 \right) - \mu^2 \star v + 2\lambda v \star \left(-\frac{\omega^2}{4\lambda}\widetilde{x}^2 + \frac{\mu^2}{4\lambda} \right) \\ &+ 2\lambda \left(-\frac{\omega^2}{4\lambda}\widetilde{x}^2 + \frac{\mu^2}{4\lambda} \right) \star v = 0. \end{aligned} \tag{61}$$

which one rewrites as

$$\frac{1}{2}\left(1 - \omega^2\right)\left(v \star \widetilde{x}^2 + \widetilde{x}^2 \star v \right) = 0. \tag{62}$$

If one now requires a non-trivial vacuum $v(x)$, then using (A.6), one has that the coefficients in the matrix basis of $v\star\widetilde{x}^2+\widetilde{x}^2\star v$ are $\frac{8}{\theta}(m+n+1)v_{mn}$ and, as a consequence, $v\star\widetilde{x}^2+\widetilde{x}^2\star v\neq 0$. This implies that the only solution of (62) is

$$\omega=1\,, \tag{63}$$

which, by (60) leads to

$$\Omega'=0\,, \tag{64}$$

as already stated above.

However, (64) considerably simplifies (56), which, once introduced in the action (54), leaves, for the π fields part, only the π^4 interaction term. Obviously, this is not physically satisfying.

Finally, let us remark that this situation is imposed by the cancellation (64), which on its turn is a consequence of the constraint (59) for a vacuum $v(x)$ (solution of the equation of motion). So it seems that the condition (59), coming from the Goldstone theorem in commutative theories, cannot be imposed for this type of models.

As stated in the beginning of this section, all these calculations are made in the case of a set of real fields ϕ and for the particular value $\Omega=1$. If one allows other values of the parameter Ω and also considers complex fields ϕ, the situation is more intricate. A possible way of approach is to combine the two constraints (i.e. the equation of motion and the masslessness (56) of the fields π) into some stronger constraint for the vacuum $v(x)$, constraint which finally has to be checked for solutions.

4 Concluding remarks

We have thus analyzed in this paper the spontaneous symmetry breaking of the non-commutative scalar model with a harmonic term, and we found that the mass value $\mu^2=\frac{4}{\theta}$ has a particular importance. For the real case, for $\mu^2<\frac{4}{\theta}$, the value $\phi(x)=0$ is the global minimum of the action. For $\mu^2\geq\frac{4}{\theta}$, the value $\phi(x)=0$ is no longer a local minimum, so the theory acquires a non-trivial vacuum; a local minimum (43) of the action was found. A further line of work is to develop the theory around this vacuum and to study its renormalizability and its renormalization group flows, as it seems to provide low values for the renormalized mass.

We have also analyzed the spontaneous symmetry breaking for a corresponding linear sigma model with N scalar fields and a harmonic term present in the action for each of these N fields. Even though this seems to be the most natural way to construct such a linear sigma model, one cannot a priori state a conclusion with respect to the renormalizability of this model. Moreover, since one has to deal with vacua that have a non-trivial dependence on the space-time position x, one can argue on the interpretation of phenomena like spontaneous symmetry breaking or the Goldstone theorem. What we have achieved in this paper is a calculation of these usual notions of commutative classical field theory in the framework of Grosse–Wulkenhaar-like models.

Finally, let us end this paper by reminding the existence of a second class of non-commutative models, called "covariant models". Here one can include the non-commutative Gross–Neveu model or the Langmann–Szabo–Zarembo model [43]. These models also were proven to be renormalizable [44] and their one-loop β-function was computed [45]. Moreover, their parametric representation was implemented in [46] (see [47] for a general review) and the Mellin representation of their Feynman amplitudes (as well as for the Grosse–Wulkenhaar model) was obtained in [48].[1]

Appendix: the matrix basis

In this appendix we present the definition and some useful properties of the matrix basis of the Moyal space. For more details, see for example [33, 41].

Let f_{mn}, $m,n\in\mathbb{N}^{\frac{D}{2}}$ be the set of Schwartz functions forming the matrix basis, to be given below. Let

$$H=\sum_{\ell=1}^{\frac{D}{2}}H_\ell,\quad H_\ell=\frac{1}{2}\left(x_{2\ell-1}^2+x_{2\ell}^2\right)\,,\ \text{for } \ell=1,\dots,\frac{D}{2} \tag{A.1}$$

Furthermore, let

$$f_{00}(x)=2^{\frac{D}{2}}\,\mathrm{e}^{-\frac{2}{\theta}H}\,. \tag{A.2}$$

This verifies

$$f_{00}\star f_{00}=f_{00}\,. \tag{A.3}$$

One also defines the operators

$$a_\ell=\frac{1}{\sqrt{2}}(x_{2\ell-1}+\mathrm{i}x_{2\ell})\,,\quad \bar{a}_\ell=\frac{1}{\sqrt{2}}(x_{2\ell-1}-\mathrm{i}x_{2\ell})\,, \tag{A.4}$$

together with

$$f_{mn}(x)=\frac{1}{\sqrt{m!n!\theta^{m+n}}}\bar{a}^{\star m}\star f_{00}\star a^{\star n}\,. \tag{A.5}$$

These functions diagonalize the Hamiltonian (A.1):

$$H\star f_{mn}=\theta\left(m+\frac{1}{2}\right)f_{mn}\,,\quad f_{mn}\star H=\theta\left(n+\frac{1}{2}\right)f_{mn}\,. \tag{A.6}$$

Some useful properties are

$$f_{mn}^\dagger=f_{nm}\,,$$
$$f_{mn}\star f_{kl}(x)=\delta_{nk}f_{ml}(x)\,. \tag{A.7}$$

[1] For some general reviews on the latest developments on the renormalizability of non-commutative quantum field theories, one may refer to [36, 49].

Finally, let us give, in $D=2$, the functions

$$f_{10}(x)=2\sqrt{\frac{2}{\theta}}(x_1-\mathrm{i}x_2)\,\mathrm{e}^{-\frac{x^2}{\theta}}\,,$$
$$f_{01}(x)=2\sqrt{\frac{2}{\theta}}(x_1+\mathrm{i}x_2)\,\mathrm{e}^{-\frac{x^2}{\theta}}\,. \tag{A.8}$$

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